Final STT2700 feuille de note.

$\sum X_i \sim N(n\mu, n\sigma^2)$

1. TCL $X \sim N(\mu, \sigma^2) \rightarrow \bar{X} \rightarrow N(\mu, \frac{\sigma^2}{n})$

$$\frac{X-\mu}{\sigma} \sim N(0,1) \quad \text{et} \quad \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

2. Condition suffisante de convergence en proba.

$$\lim_{n\to\infty} E(X) \to c. \qquad \lim_{n\to\infty} var(X) = 0.$$

$$\Rightarrow X \xrightarrow{P} c$$

3. Colloraire de (2)

$$\lim_{n\to\infty} E(X_n - X) = 0. \quad var(X_n - X) \xrightarrow[n\to\infty]{} 0.$$

$$\Rightarrow X_n \xrightarrow{P} X.$$

4. Convergence en moyenne d'ordre K.

$$\lim_{n\to\infty} E(|X_n - X|^K) = 0. \Rightarrow X_n \xrightarrow{L^K} X \text{ ou } X_n \xrightarrow{L^K} X$$

5. $X_n \xrightarrow{L_2} X \Leftrightarrow E(X_n - X) \to 0, \ var(X_n - X) \to 0.$

6. Convergence en Loi.

$$X_n \xrightarrow{Loi} X \Leftrightarrow \lim_{n\to\infty} F_n(x) = F(x)$$

Si $\lim_{n\to\infty} M_n(t) = M(t)$, alors $\lim_{n\to\infty} F_n(x) = F(x)$

7. $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x \quad , \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad \lim_{n\to\infty} 1 - \frac{1}{\left(1+\frac{x}{n}\right)^n} = 1-e^{-x}$

$$\int_0^\infty e^{-y}\,dy = 1. \quad \int_0^y e^{-v}\,dv = 1-e^{-y}$$

$$\int_0^1 t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

8. $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{Xt} \cdot e^{Yt}) = E(e^{Xt}) \cdot E(e^{Yt})$

$$= M_X(t) \cdot M_Y(t)$$

$M_{a+bX}(t) = E(e^{t(a+bX)}) = E(e^{at} \cdot e^{bXt}) =$

$E(e^{at}) \cdot E(e^{bXt}) = e^{at} \cdot M_X(bt)$

9. $X \sim Exp(\lambda)$, $E(X^k) = \frac{k!}{\lambda^k}$

10. $a_n(X_n - c) \xrightarrow{Loi} X$ avec $a_n \uparrow$, $X \to N(0, \sigma^2)$

$\Rightarrow \sqrt{n}(g(X_n) - g(c)) \xrightarrow{Loi} N(0, \sigma^2 \cdot g'(c)^2)$

11. $X_1 \ldots X_n$ $X_i \sim N(\mu_i, \sigma_i^2)$

$Y = \sum_{i=1}^{n} a_i X_i + b \sim N\left(\sum_{i=1}^{n} a_i \mu_i + b, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$

12. $X \sim N(\mu, \sigma^2)$. $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. $Z^2 \sim \chi^2_1$

13. $Y_1, \ldots Y_n$ iid χ^2_1.

$\sum_{i=1}^{n} Y_i \sim \chi^2_n \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

14. $Z \sim N(0,1)$ $U \sim \chi^2_n$. $T = \frac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$

15. $U \sim \chi^2_m$ $V \sim \chi^2_n$ $W = \frac{U/m}{V/n} \sim F_{m,n}$

16. $\frac{S^2 \cdot (n-1)}{\sigma^2} \sim \chi^2_{n-1}$

17. EVM : estimateur de vraisemblance.

$X \sim Poi(\theta = \lambda)$ $\quad \hat{\lambda} = \bar{X}$

$X \sim Bernoulli(\theta = p)$ $\quad \hat{p} = \bar{X}$

$X \sim Binomial(n, p)$ $\quad \hat{p} = \frac{X}{n}$ (fréquence relative)

$X \sim N(\mu, \sigma^2)$ $\quad \begin{cases} \hat{\mu} = \bar{X} \\ \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n} S^2 \end{cases}$

$X \sim U[0, \theta]$ $\quad \hat{\theta} = X_{(n)}$

$X \sim exp(\lambda = \frac{1}{\theta})$ $\quad \hat{\theta} = \bar{X} = \frac{1}{\hat{\lambda}}$

$$\boxed{\begin{array}{l} X \sim Gamma(\alpha, \lambda) \\ \hat{\lambda} = \dfrac{\bar{X}}{\hat{\sigma}^2} \\ \hat{\alpha} = \bar{X}^2 / \hat{\sigma}^2 \end{array}}$$

méthode de moment

18. $X \sim N(\mu, \sigma^2)$.

$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$.

19. $Biais(\hat{\theta}) = E(\hat{\theta}) - \theta$.

Sans biais $\Leftrightarrow$ $E(\hat{\theta}) = \theta$.

$E(\bar{X}) = \mu$. $\quad E(S^2) = \sigma^2$. $\quad$ sans biais.

20. $X \sim Bin(n, p)$ — $E(X) = np$, $var(X) = np(1-p)$

$E(\bar{X}) = \theta$. $X \sim \text{Bernoulli}(\theta = p)$ aussi

l'estimateur de p $\hat{p} = \frac{X}{n}$.

$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n} E(X) = \frac{1}{n} \cdot np = p$. (sans biais)

$var(\hat{p}) = var\left(\frac{X}{n}\right) = \frac{1}{n^2} var(X) = \frac{1}{n^2} n(1-p)p = \frac{(1-p)p}{n}$.

21. $X \sim Poi(\lambda)$ — $E(X) = \lambda$, $var(X) = \lambda$ $\quad E(\bar{X}) = \lambda$, $var(\bar{X}) = \frac{\lambda}{n}$.

l'estimateur de $\lambda = \bar{X}$

22. $X \sim U[0, \theta]$ — $E(X) = E(\bar{X}) = \frac{b-a}{2}$, $var(X) = (b-a)^2/12$

(1) $\hat{\theta} = 2\bar{X}$ $\quad E(\hat{\theta}) = 2E(\bar{X}) = \theta$. sans biais.

(2) $\hat{\theta} = X_{(n)}$ $\quad E(X_{(n)}) = \int_0^\theta \left(1 - F_{X_{(n)}}(x)\right) dx = \frac{n}{n+1}\theta$.

$\text{biais}(\hat{\theta}) = \frac{1}{n+1}\theta$.

23. $X \sim Poi(\lambda)$ $\quad T(X)$.

$E(T) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} \cdot T(x)$

FGM. $S \sim Poi(n\lambda)$

si $T(X) = e^{-\bar{X}}$, $E(T(X)) = E(e^{-\bar{X}}) = \boxed{E(e^{-S/n}) = M_S(-\frac{1}{n})}$

FGM de S?

24. L'estimateur $\hat{\theta}$ est asymptotiquement sans biais

Si $\lim_{n \to \infty} E(\hat{\theta}) = \theta$.

25. Si $\hat{\theta}$ est asymptotiquement sans biais,

et $\lim_{n \to \infty} \mathrm{var}(\hat{\theta}) = 0$. $\Rightarrow$ $\hat{\theta}$ est convergent pour θ.

26. $EQM(\hat{\theta}) = \mathrm{var}(\hat{\theta}) + [\mathrm{biais}(\hat{\theta})]^2$

$\mathrm{biais}(\hat{\theta}) = E(\hat{\theta}) - \theta$.

27. $X_1, \ldots X_n \sim \mathrm{Bernoulli}\ (\theta = p)$. $T = \sum_{i=1}^{n} X_i$ est exhaustive.

$P(X_1, \ldots X_n | T=t) = \dfrac{P(X_1, \ldots X_n, T=t)}{P(T=t)} = \dfrac{1}{\binom{n}{t}}$ ne dépend pas de θ.

28. factorisation de Fisher-Neyman.

$f(x_1, \ldots x_n | \theta) = \prod_{i=1}^{n} f(x_i | \theta) = g(T | \theta) \cdot h(x_1, \ldots x_n)$

29. Région de vraisemblance.

$C(x) = \{ \theta \in \Theta : L(\theta | x) \geq c \}$.

30. Si $\sqrt{n}(X_n - \mu) \xrightarrow{Loi} N(0, \sigma^2)$ et

$$E(X_n) = \mu \text{ et } var(X_n) = \frac{\sigma^2}{n}.$$

$$E(g(X_n)) \approx g(\mu) + \frac{1}{2}|g''(\mu)| \cdot \frac{\sigma^2}{n}.$$

$$|\text{biais}(g(X_n))| = \frac{1}{2}|g''(\mu)| \cdot \frac{\sigma^2}{n}$$

31. Information de Fisher

$$I(\theta) = E\Big[\underbrace{\Big(\frac{\partial}{\partial\theta} \ln f(x|\theta)\Big)^2}_{g(x)}\Big] = \int g(x) \cdot f(x)\, dx.$$

$$I(\theta) = var\Big[\frac{\partial}{\partial\theta} \ln f(x|\theta)\Big]$$

$$I(\theta) = -E\Big[\frac{\partial^2}{\partial\theta^2} \ln f(x|\theta)\Big]$$

$X \sim exp(\lambda)$ $\quad I(\lambda) = \frac{1}{\lambda^2}$

$X \sim poi(\lambda)$ $\quad I(\lambda) = \frac{1}{\lambda}$

$X \sim N(\theta, \sigma^2)$ $\quad I(\theta) = \frac{1}{\sigma^2}$

$X \sim Bernoulli(\theta)$ $\quad I(\theta) = \frac{1}{\theta(1-\theta)}$

$X \sim exp(\frac{1}{\theta})$ $\quad I(\theta) = \frac{1}{\theta^2}$

$$\boxed{\begin{array}{l} \hat{\theta} \sim N\left(\theta, \frac{1}{nI(\theta)}\right) \\ var(\hat{\theta}) \approx \frac{1}{nI(\theta)} \end{array}}$$

variance asymptotique.

32. Intervalles de confiance

$$IC_{1-\alpha}(\mu) = \bar{X} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

$$IC_{1-\alpha}(\sigma^2) = \left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1,\frac{\alpha}{2}}} , \frac{n\hat{\sigma}^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right)$$

$$\boxed{IC_{1-\alpha}(\theta) = \hat{\theta} \pm z_{\frac{\alpha}{2}} \cdot \frac{1}{\sqrt{nI(\theta)}}}$$

Bernoulli: $\hat{\theta} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sqrt{\theta(1-\theta)}}{\sqrt{n}}$

exponentiel (λ) $\hat{\lambda} \pm z_{\frac{\alpha}{2}} \frac{\lambda}{\sqrt{n}}$, $\hat{\lambda} = \frac{1}{\bar{X}}$.

$$IC(\lambda) = \frac{1}{\bar{X}} \pm z_{\frac{\alpha}{2}} \frac{1}{\sqrt{n}\,\bar{X}}$$

32. Transformation stabilisant la variance.

Note Chap 8 Note 3 P16 118

33. Borne Cramer-Rao

$$\mathrm{var}(T) \geq \frac{\tau'(\theta)^2}{nI(\theta)}$$ T est un estimateur de $\tau(\theta)$

Si $\tau(\theta) = \theta$.

$$\mathrm{var}(T) \geq \frac{1}{nI(\theta)}$$

34. Famille exponentielle

$$f(x|\theta) = c(\theta) h(x) \exp\left(\sum_{i=1}^{k} q_i(\theta) t_i(x) \right)$$

35.

Loi à priori $f(\theta)$ $(h(\theta))$

Loi à postériori $\Theta | X$

discrète:

$$P(\Theta = \theta | X = x) = \frac{P(X = x | \Theta = \theta) \cdot P(\Theta = \theta)}{\sum_{\theta \in \Theta} P(X = x | \Theta = \theta)}$$

Continue:

$$f(\theta|X) = \frac{f(x,\theta)}{f(x)} = \frac{\prod_{i=1}^{n} f(x_i|\theta)\cdot f(\theta)}{\int_R f(x,\theta)\, d\theta}$$

$$= \frac{\prod_{i=1}^{n} f(x_i|\theta) f(\theta)}{\int_R \prod_{i=1}^{n} f(x_i|\theta) f(\theta)\, d\theta}$$

36. Gamma.

$X \sim Gamma(r, \lambda)$. $Y = aX$, $Y \sim Gamma(r, \frac{\lambda}{a})$.

$Gamma(\frac{v}{2}, \frac{1}{2}) \sim \chi^2_v$

37. $X \sim Gamma(\alpha, \lambda)$

$$E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$$

38. Le test de N-P

$$\frac{\prod_{i=1}^{n} f_0(x)}{\prod_{i=1}^{n} f_1(x)} < C.$$

39. α : $P(R \mid H_0 \text{ vraie})$

$1-\beta$: $P(R \mid H_0 \text{ fausse})$

β : $P(\neg R \mid H_0 \text{ fausse})$

40. Le test le plus puissant.

$$\frac{L(\theta_0 \mid x)}{L(\theta_1 \mid x)} = \frac{\prod_{i=1}^{n} f(x_i \mid \theta_0)}{\prod_{i=1}^{n} f(x_i \mid \theta_1)} < C.$$

Uniformément : R ne dépend pas de θ_1.

41. le test du rapport de vraisemblance.

$$\Lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)} < C.$$

$\leftarrow \theta_0$

$\leftarrow \hat{\theta}_{EVM}$.

42. Région critique du test du rapport de vraisemblance au niveau asymptotique α.

$\alpha = P(R \mid H_0)$ penser à TCL.