# Chapter 7. Correlated Errors

In this chapter, we examine the case where

$E(\varepsilon\varepsilon') = \sigma^2\Omega$ could be non-diagonal.

i.e. Some $E(\varepsilon_j \varepsilon_i)$ may be non-zero even when $i \neq j$.

under Gauss-Markov, $Cov(y) = \sigma^2 I$.

(1) $E(\hat{\beta}) = \beta$

(2) $$Cov(\hat{\beta}) = (X'X)^{-1}X' \, Cov(y) \, X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1} X' \Omega X (X'X)^{-1}.$$

$$\neq \sigma^2 (X'X)^{-1}.$$

## Generalized Least Squares: Case when $\Omega$ is known

Consider the usual regression model

$y = X\beta + \varepsilon$. $E(\varepsilon) = 0$. $E(\varepsilon\varepsilon') = \sigma^2\Omega$

- $y$ is the response vector of $n$ observations.
- $X$ is an $n \times (k+1)$ matrix of known constants
- $\beta$ is the $(k+1)$ vector of unknown regression parameters

· $\Omega$ is a known symmetric, positive definite matrix of order $n$.

Under these conditions, a preferred estimate of $\beta$ is the generalized least squares estimator:

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

$$y^{(\Omega)} = \Omega^{-\frac{1}{2}}y. \quad X^{(\Omega)} = \Omega^{-\frac{1}{2}}X. \quad \varepsilon^{(\Omega)} = \Omega^{-\frac{1}{2}}\varepsilon.$$

$$y^{(\Omega)} = X^{(\Omega)}\beta + \varepsilon^{(\Omega)}; \; E(\varepsilon^{(\Omega)}) = 0; \; cov(\varepsilon^{(\Omega)}) = \sigma^2 I.$$

Estimate of parameters

$$y = X\beta + \varepsilon. \quad E(\varepsilon) = 0. \quad cov(\varepsilon) = \sigma^2\Omega.$$

(1) $\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \beta + A\varepsilon,$

$A = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}.$

(2) $cov(\hat{\beta}_{GLS}) = \sigma^2(X'\Omega^{-1}X)^{-1}.$ $\quad E(\hat{\beta}_{GLS}) = \beta.$

(3) $s^2 = \dfrac{e^{(\Omega)'}e^{(\Omega)}}{n-k-1}.$

where: $e^{(\Omega)} = \left[ I - X^{(\Omega)} \left( X^{(\Omega)'} X^{(\Omega)} \right)^{-1} X^{(\Omega)'} \right] y^{(\Omega)}$

proof:

(1) $S = \left( y^{(\Omega)} - X^{(\Omega)}\beta \right)' \left( y^{(\Omega)} - X^{(\Omega)}\beta \right)$

$$= y'^{(\Omega)} y^{(\Omega)} - 2\left( X'^{(\Omega)} y^{(\Omega)} \right)' \beta + 2\beta' \left( X'^{(\Omega)} X^{(\Omega)} \right) \beta$$

$$\frac{\partial S}{\partial \beta} = -2\left( X'^{(\Omega)} y^{(\Omega)} \right) + 2\left( X'^{(\Omega)} X^{(\Omega)} \right) \beta$$

$$\left. \frac{\partial S}{\partial \beta} \right|_{\beta = \hat\beta_{GLS}} = 0 \Rightarrow \hat\beta_{GLS} = \left( X'\Omega^{-1}X \right)^{-1} X'\Omega^{-1} y$$

(2) $$\operatorname{cov}(\hat\beta_{GLS}) = \left( X'\Omega^{-1}X \right)^{-1} X' \Omega^{-1} \overbrace{\operatorname{cov}(y)}^{\sigma^2\Omega} \Omega^{-1} X \left( X'\Omega^{-1}X \right)^{-1}$$

$$= \left( X'\Omega^{-1}X \right)^{-1} X' \Omega^{-1} \sigma^2 \Omega \Omega^{-1} X \left( X'\Omega^{-1}X \right)^{-1}.$$

$$= \sigma^2 \left( X'\Omega^{-1}X \right)^{-1} X' \underbrace{\Omega^{-1}\Omega}_{I} \Omega^{-1} X \left[ \left( X'\Omega^{-1}X \right)^{-1} \right]'$$

$$= \sigma^2 \left( X'\Omega^{-1}X \right)^{-1} \underbrace{X'\Omega^{-1}X \left( X'\Omega^{-1}X \right)^{-1}}_{I.}$$

$$= \sigma^2 \left( X'\Omega^{-1}X \right)^{-1}.$$

(3) $E(e'^{(\Omega)} e^{(\Omega)}) = E(y'^{(\Omega)} M'^{(\Omega)} M^{(\Omega)} y^{(\Omega)})$

où $\boxed{e^{(\Omega)} = y^{(\Omega)} - \hat{y}^{(\Omega)} = [I_n - X^{(\Omega)}(X'^{(\Omega)} X^{(\Omega)})^{-1} X'^{(\Omega)}] y^{(\Omega)}}$

$= M^{(\Omega)} y^{(\Omega)}$

$\Rightarrow E(e'^{(\Omega)} e^{(\Omega)}) = \boxed{E(y'^{(\Omega)} M^{(\Omega)} y^{(\Omega)})}$ car $M'^{(\Omega)} M^{(\Omega)} = M^{(\Omega)}$

$\boxed{= tr[M^{(\Omega)} E(y'^{(\Omega)} y^{(\Omega)})]}$ $E(\varepsilon'^{(\Omega)} \varepsilon^{(\Omega)}) = \sigma^2 I_n$

$= tr[M^{(\Omega)} \sigma^2 I_n] = \sigma^2 (n-K-1)$

$\Rightarrow E(s^2) = E\left(\frac{e'^{(\Omega)} e^{(\Omega)}}{n-K-1}\right) = \frac{1}{n-K-1} E(e'^{(\Omega)} e^{(\Omega)}) =$

$\frac{1}{n-K-1} \sigma^2 (n-K-1) = \sigma^2$

Theorem: Distribution of estimators.

$y = X\beta + \varepsilon. \quad E(\varepsilon) = 0. \quad Cov(\varepsilon) = \sigma^2 \Omega$

(1) $K = \frac{(n-K-1)s^2}{\sigma^2} \sim \chi^2_{n-K-1}$.

(2) $\hat{\beta}_{GLS}$ and $s^2$ are independant.

(3) $F = \dfrac{(\hat{\beta}_{GLS} - \beta)' (X' \Omega^{-1} X) (\hat{\beta}_{GLS} - \beta)}{s^2} \sim F_{k+1,\, n-k-1}$.

If we wish to test the hypothesis:

$H: c\beta - d = 0.$ vs $A: c\beta - d \neq 0.$

$$\boxed{\begin{array}{l}\text{H is rejected if} \\ F = \dfrac{1}{m s^2} (c\hat{\beta}_{GLS} - d)' [c (X' \Omega^{-1} X)^{-1} c']^{-1} \\ \qquad (c\hat{\beta}_{GLS} - d) \geq F_{m,\, n-k-1;\, \alpha}\end{array}}$$

$$s^2 = \frac{e^{(\Omega)'} e^{(\Omega)}}{n-k-1.} = (n-k-1)[y' \Omega^{-1} y - \hat{\beta}'_{GLS} X' \Omega^{-1} y].$$

proof:

(1) $s^2 = \dfrac{e'^{(\Omega)} e^{(\Omega)}}{n-k-1}$ où $e^{(\Omega)} = M^{(\Omega)} \varepsilon^{(\Omega)}$

$\varepsilon^{(\Omega)} \sim N(0, \sigma^2 I_n)$

$$k = \frac{(n-k-1) s^2}{\sigma^2} = \frac{\varepsilon'^{(\Omega)} M'^{(\Omega)} M^{(\Omega)} \varepsilon^{(\Omega)}}{\sigma^2} = \frac{\varepsilon'^{(\Omega)} M^{(\Omega)} \varepsilon^{(\Omega)}}{\sigma^2}$$

$$\sim \chi^2_{tr(M^{(\Omega)})} \quad \text{où} \quad tr(M^{(\Omega)}) = n-(k+1)$$

(3) $\hat{\beta}_{GLS} \sim N_{k+1}(\beta, \sigma^2 V) \quad V = (X'\Omega^{-1}X)^{-1}.$

$$\Rightarrow Z = \frac{V^{-\frac{1}{2}}(\hat{\beta}_{GLS} - \beta)}{\sigma} \sim N_{k+1}(0, I_{k+1})$$

Let $K_1 = Z'Z = \dfrac{(\hat{\beta}_{GLS}-\beta)' V^{-1} (\hat{\beta}_{GLS}-\beta)}{\sigma^2} \sim \chi^2_{k+1}.$

$$K_2 = \frac{(n-k-1)s^2}{\sigma^2} \sim \chi^2_{n-k-1}$$

$K_1$ $K_2$ independant,

$$F = \frac{K_1/(k+1)}{K_2/(n-k-1)} = \frac{(\hat{\beta}_{GLS}-\beta)' X' \Omega^{-1} X (\hat{\beta}_{GLS}-\beta)}{(k+1)s^2} \sim F_{k+1,\, n-k-1}.$$

ou

$$F = \frac{(C\hat{\beta}_{GLS}-\beta)' X' \Omega^{-1} X (\hat{\beta}_{GLS}-\beta)}{(k+1)s^2} \sim F_{k+1,\, n-k-1}.$$

avec $C = I_{k+1}$. $\quad C$ dim $m \times (k+1)$

$$C\hat{\beta}_{GLS} \sim N_m\left(C\beta,\ \sigma^2 C(X'\Omega^{-1}X)^{-1}C'\right)$$

$$\Rightarrow C\hat{\beta}_{GLS} - C\beta \sim N_m(0,\ \sigma^2 G),\quad G = C(X'\Omega^{-1}X)^{-1}C'$$

$$\Rightarrow z = \frac{G^{-\frac{1}{2}}(C\hat{\beta}_{GLS} - C\beta)}{\sigma} \sim N_m(0, I_m)$$

Let $K_1 = z'z = \dfrac{(C\hat{\beta}_{GLS} - C\beta)'\, G^{-1}\, (C\hat{\beta}_{GLS} - C\beta)}{\sigma^2}$

$$\sim \chi^2_m$$

$$\Rightarrow F = \frac{K_1 / m}{K_2 / (n-K-1)} = \frac{(C\hat{\beta}_{GLS} - C\beta)'\overbrace{\left[C(X'\Omega^{-1}X)^{-1}C'\right]^{-1}}^{G^{-1}}(C\hat{\beta}_{GLS} - C\beta)}{ms^2}$$

$$\sim F_{m,\, n-K-1}.$$

$H_0$: $C\beta = d$.

$$F = \frac{(C\hat{\beta}_{GLS} - d)'\left[C(X'\Omega^{-1}X)^{-1}C'\right]^{-1}(C\hat{\beta}_{GLS} - d)}{ms^2} \sim F_{m,\, n-K-1;\, \alpha}$$

## Estimated Generalized Least Squares: unknown $\Omega$.

The matrix is not usually known and needs to be estimated. If $\hat{\Omega}$ is an estimated of $\Omega$, then

$$\hat{\beta}_{EGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}y$$

has been called an estimated generalized least squares estimate.

General asymptotic properties are available:

(1) $\hat{\beta}_{GLS}$ and $\hat{\beta}_{EGLS}$ are both consistent and have the same asymptotic distribution.

(2) Both estimates are asymptotically normal with mean $\beta$ and covariance matrix

$$\frac{\sigma^2\Phi^{-1}}{n}$$

(3) $$\frac{\hat{\beta}_{GLS} - \hat{\beta}_{EGLS}}{\sqrt{n}} \xrightarrow{P} 0$$

(4) Under some further conditions,

$$\hat{\sigma}^2 = \frac{(y - X\hat{\beta}_{EGLS})' \hat{\Omega}^{-1} (y - X\hat{\beta}_{EGLS})}{n-k-1} \xrightarrow{P} \sigma^2$$

special case: error variance unequal and unknown

A special case where we need to consider empirical estimation of $\Omega$ occurs when the non-diagonal elements of $\Omega$ are zero and the diagonal elements are unknown.

i.e. $\sigma^2 \Omega = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \cdots \sigma_n^2)$

and the $\sigma^2$ are unknown.

This is case of heteroscedasticity with unknown variances.

## Estimation procedure

- Suppose we apply ordinary least squares and obtain the vector $e$ of residuals $e_i$.

  It is reasonable to base any estimation of $\sigma_i^2$ on these residuals. $e = y - X\hat{\beta}$

- A standard method consists of considering

$$\boxed{e^{(2)} = (e_1^2, e_2^2, \dots e_n^2)'}$$

Since $e = M\varepsilon$, $M = I - X(X'X)^{-1}X'$,

$$e_i = \sum_{j=1}^{n} m_{ij} \cdot \varepsilon_j, \quad E[\varepsilon_i \varepsilon_j] = \sigma_j^2 \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$E(e_i^2) = \sum_{j=1}^{n} m_{ij} \sigma_j^2$$

Let $M^{(2)} = (m_{ij}^2)$ and $\sigma^{(2)} = (\sigma_1^2, \sigma_2^2, \dots \sigma_n^2)'$

$$\boxed{E(e^{(2)}) = M^{(2)} \sigma^{(2)}}$$

Replacing $E(e^{(2)})$ by its estimates $e^{(2)}$ and $\sigma^{(2)}$. by its estimates $\hat{\sigma}^{(2)} = (\hat{\sigma}_1^2, \hat{\sigma}_2^2, \dots, \hat{\sigma}_n^2)$

We get $\boxed{e^{(2)} = M^{(2)} \cdot \hat{\sigma}^{(2)}}$

We can solve this set of equations to get the required estimates $\hat{\sigma}_i^2$.

These estimates are also known to be MINQUE.

⊛ A major difficulty with these estimates is that some $\hat{\sigma}_i^2$ can turn out to be negative.

Although, it is not desirable to estimate individual $\sigma_i^2$ in this way, the method can be useful if there is a relationship among the $\sigma_i^2$'s

assume that $\boxed{\sigma_i^2 = z_i' \alpha}$, $\sigma^2 \Omega = \text{diag}(z_1'\alpha, \dots z_n'\alpha)$

$z_i$ is an $m$-dimensional known vector

$\alpha$ is a vector of parameters.

Let $Z' = (Z_1, \dots Z_n)$, $E(e^{(2)}) = M^{(2)} Z\alpha$,

which prompts the estimation of $\alpha$ as

$$\hat{\alpha} = (Z' M^{(2)\prime} M^{(2)} Z)^{-1} Z' M^{(2)} e^{(2)}$$

$\hat{\sigma}_i^2 = z_i' \hat{\alpha}$

which in turn can be used to estimate $\sigma_i^2$'s

Since $\boxed{(M^{(2)})^{-1} E(e^{(2)}) = Z\alpha}$.

another estimate of $\alpha$ is

$$\hat{\alpha} = (Z'Z)^{-1} Z' (M^{(2)})^{-1} e^{(2)}$$

Yet another alternative (MINQUE, Froehlich, 1973)

$$\hat{\alpha} = (Z' M^{(2)} Z)^{-1} Z' e^{(2)}$$

ex: $y_i = \beta_0 + (\beta_1 + \delta_i) x_i + \varepsilon_i$

$\Rightarrow y_i = \beta_0 + \beta_1 x_i + \delta_i x_i + \varepsilon_i$

Suppose that $\delta_i$ i.i.d $E(\delta_i) = 0$, $var(\delta_i) = \varphi^2$

$\varepsilon_i$ i.i.d $E(\varepsilon_i) = 0$, $var(\varepsilon_i) = \sigma^2$

$\varepsilon$ and $\delta$ are independant.

$y_i = \beta_0 + \beta_1 x_i + \gamma_i$, $E(\gamma_i) = 0$,

$$\text{var}(\gamma_i) = \varphi^2 x_i^2 + \sigma^2$$

$$= \underbrace{\begin{bmatrix} 1 & x_i^2 \end{bmatrix}}_{z_i'} \underbrace{\begin{bmatrix} \sigma^2 \\ \varphi^2 \end{bmatrix}}_{\alpha}.$$

obtain $\hat{\alpha} \to$ OLS $\to$ $w_i = \dfrac{1}{\text{var}(\gamma_i)} = \dfrac{1}{z_i'\hat{\alpha}}$

$\to \hat{\beta}$ EGLS

## Serial Correlation.*

Frequently when the observations $y_i$ are taken over successive time intervals, the $\varepsilon_i$'s are correlated. This type of correlation is called serial correlation. We consider the particular case when the $\varepsilon_i$'s follow a first order autoregressive process:

$$\varepsilon_t = \rho \varepsilon_{t-1} + \eta_t.$$

$y = X\beta + \varepsilon$.    $E(\varepsilon) = 0$.    $\text{Cov}(\varepsilon^2) = \sigma^2 \Omega$

where $|\rho| < 1$ and $\forall\, t = 1, \dots n$. $\eta_t$'s are independent and identically distributed with mean 0 and variance $\sigma^2$. $\mathrm{Cov}(\eta_i, \eta_j) = \sigma^2 \delta_{ij}$

$E(\varepsilon_t) = 0$. $\mathrm{var}(\varepsilon_t) = \dfrac{\sigma^2}{1-\rho^2}$

$$\sigma^2 \Omega = \frac{\sigma^2}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & & & & \vdots \\ \vdots & & & & \vdots \\ \rho^{n-1} & \cdots & & & 1 \end{bmatrix}$$

$$\hat{\rho} = \frac{\sum_{t=2}^{n} e_t e_{t-1}}{\sum_{t=2}^{n} e_{t-1}^2} \longrightarrow \hat{\Omega}$$

$$\hat{\beta}_{EGLS} = (X' \hat{\Omega}^{-1} X)^{-1} X' \hat{\Omega}^{-1} y.$$

## 7.5 The Growth Curve Model

When nothing is known of the form of $\Omega$, it is still possible to estimate it, if the experiment can be replicated an adequate number of times, i.e., if we have the model

$$\boldsymbol{y}_t = X\boldsymbol{\beta} + \boldsymbol{\epsilon}_t$$

where $X$ is an $m \times p$ matrix, $t = 1, 2, \ldots, M$, $\mathrm{E}(\boldsymbol{\epsilon}_t) = \mathbf{o}$ and $\mathrm{cov}(\boldsymbol{\epsilon}_t) = \Omega$. Here $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_M$ are independent $m$ dimensional vectors and $\sigma^2$ has been absorbed in $\Omega$.

Let $\bar{\boldsymbol{y}} = M^{-1}\sum_{t=1}^{M} \boldsymbol{y}_t$ and $\bar{\boldsymbol{\epsilon}} = M^{-1}\sum_{t=1}^{M} \boldsymbol{\epsilon}_t$. Then

$$\bar{\boldsymbol{y}} = X\boldsymbol{\beta} + \bar{\boldsymbol{\epsilon}}$$

where $\mathrm{E}(\bar{\boldsymbol{\epsilon}}) = \mathbf{o}$ and $\mathrm{cov}(\bar{\boldsymbol{y}}) = \mathrm{cov}(\bar{\boldsymbol{\epsilon}}) = M^{-1}\Omega$. Hence, the generalized least squares estimate of $\boldsymbol{\beta}$ is given by

$$\boldsymbol{b}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\bar{\boldsymbol{y}}.$$

However, $\Omega$ is not known. But since $\mathrm{E}(\boldsymbol{y}_t - \bar{\boldsymbol{y}}) = 0$, for all $t = 1, \ldots, M$, an unbiased estimator of $\Omega$ is given by

$$\hat{\Omega} = (M-1)^{-1}\sum_{t=1}^{M}(\boldsymbol{y}_t - \bar{\boldsymbol{y}})(\boldsymbol{y}_t - \bar{\boldsymbol{y}})'.$$

Therefore, an estimated generalized least squares estimate of $\boldsymbol{\beta}$ is

$$\boldsymbol{b}_{EGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}\bar{\boldsymbol{y}}.$$

Under the assumption that $\boldsymbol{\epsilon}_t$ is multivariate normal, it can be shown that this $\boldsymbol{b}_{EGLS}$ is an unbiased estimator (see Exercise 7.3).

Under normality, the hypothesis $H : C\boldsymbol{\beta} = 0$ against $A : C\boldsymbol{\beta} \neq 0$ (where $C$ is $r \times p$ dimensional with $r \leq p$) is rejected if

$$\frac{M-r-m+p}{(M-1)r}\,\frac{\boldsymbol{b}'_{EGLS}C'(CEC')^{-1}C\boldsymbol{b}_{EGLS}}{1+(M-1)^{-1}T^2} > F_{r,M-r-m+p,\alpha}$$

where $E = (X'\hat{\Omega}^{-1}X)^{-1}$, $T^2 = M\bar{\boldsymbol{y}}'G'(G\hat{\Omega}G')^{-1}G\bar{\boldsymbol{y}}$ and $G : (m-p) \times m$ is such that $GX = 0$. Alternatively, if it is inconvenient to find a suitable $G$, one could use $T^2 = M\bar{\boldsymbol{y}}'[\hat{\Omega}^{-1} - \hat{\Omega}^{-1}X(X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}]\bar{\boldsymbol{y}}$. (This result is obtained with the help of Lemma A.1, p. 279, of Appendix A).

**Example 7.1**
Exhibit 7.1 shows dental measurements for girls from 8 to 14 years old. Each measurement is the distance, in millimeters, from the center of the pituitary to the ptery-maxillary fissure. Suppose we wish to relate these measurements to age and write our model as

$$y_{ts} = \beta_0 + \beta_1 x_{t1} + \epsilon_{ts},$$

where $x_{t1} = \text{age} - 11$. Then

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}'.$$

Clearly, for the same subject $s$ the $m = 4$ measurements $y_t$ are not independent and

$$\text{cov}(\boldsymbol{\epsilon}_s) = \Omega, \text{ where } \boldsymbol{\epsilon}_s = (\epsilon_{1s}, \ldots, \epsilon_{4s})'.$$

However, we have $M = 11$ replications of the experiment.

| Age | Subjects | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| in Years | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 8 | 21.0 | 21.0 | 20.5 | 23.5 | 21.5 | 20.0 | 21.5 | 23.0 | 20.0 | 16.5 | 24.5 |
| 10 | 20.0 | 21.5 | 24.0 | 24.5 | 23.0 | 21.0 | 22.5 | 23.0 | 21.0 | 19.0 | 25.0 |
| 12 | 21.5 | 24.0 | 24.5 | 25.0 | 22.5 | 21.0 | 23.0 | 23.5 | 22.0 | 19.0 | 28.0 |
| 14 | 23.0 | 25.5 | 26.0 | 26.5 | 23.5 | 22.5 | 25.0 | 24.0 | 21.5 | 19.5 | 28.0 |

EXHIBIT 7.1: Data on Dental Measurements
SOURCE: Pothoff and Roy (1964). Reproduced from *Biometrika* with the permission of Biometrika Trustees.

Since $\bar{\boldsymbol{y}} = (21.2, 22.2, 23.1, 24.1)'$, an estimate of $\Omega$ is given by $\hat{\Omega} = \sum_{i=1}^{11}(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})'/10$

$$= \begin{pmatrix} 4.51 & 3.36 & 4.43 & 4.36 \\ 3.36 & 3.62 & 4.02 & 4.08 \\ 4.33 & 4.03 & 5.59 & 5.47 \\ 4.36 & 4.08 & 5.47 & 5.94 \end{pmatrix}.$$

Hence

$$\boldsymbol{b}_{EGLS} = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}\bar{\boldsymbol{y}} = \begin{pmatrix} 22.70 \\ 0.482 \end{pmatrix}.$$

Suppose we wish to test the hypothesis that the linear term is zero. That is, $H : \beta_1 = 0$ against $A : \beta_1 \neq 0$. In this case, $C = (0, 1)$, $r = 1$, $p = 2$ and

$$E = (X'\hat{\Omega}^{-1}X)^{-1} = \begin{pmatrix} 3.807 & 0.160 \\ -0.160 & 0.160 \end{pmatrix}$$

and $C\,\mathrm{E}C' = 0.045$. The matrix

$$G = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

is such that $GX = 0$. Therefore, $T^2 = 11\bar{\boldsymbol{y}}'G'(G\hat{\Omega}G')^{-1}G\bar{\boldsymbol{y}} = 0.11$ and

$$F = \frac{M - r - m + p}{(M-1)r}\,\frac{\boldsymbol{b}'_{EGLS}C'(CEC')^{-1}C\boldsymbol{b}_{EGLS}}{1 + T^2/10} = 45.94.$$

Therefore, at 1 and 8 degrees of freedom, we reject the hypothesis at a 5 per cent level. ∎