Chapitre VI

6.1.9.

$$\iint_R \sqrt{2}\, dA \quad , R = \{(x,y) \mid 2 \le x \le 6, -1 \le y \le 5\}.$$

$$= \int_{y=-1}^{5} \int_{x=2}^{6} \sqrt{2}\, dx\, dy = \int_{y=-1}^{5} [\sqrt{2}x]_2^6\, dy = \int_{y=-1}^{5} 4\sqrt{2}\, dy.$$

$$= [4\sqrt{2}y]_{-1}^{5} = 24\sqrt{2}.$$

6.1.11.

$$\iint_R (4-2y)\, dA, \quad R = [0,1] \times [0,1].$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} (4-2y)\, dx\, dy = \int_{y=0}^{1} [4x-2xy]_0^1\, dy = \int_{y=0}^{1} (4-2y)\, dy$$

$$= [4y - y^2]_0^1 = (4-1) - 0 = 3.$$

6.1.17.

$$\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy = \int_0^1 \left[\frac{x^2}{2} + e^{-y} \cdot x\right]_1^2 dy. =$$

$$\int_0^1 \left(2 + 2e^{-y} - \frac{1}{2} - e^{-y}\right) dy = \int_0^1 \left(\frac{3}{2} + e^{-y}\right) dy.$$

$$= \left[\frac{3}{2}y - e^{-y}\right]_0^1 = \left(\frac{3}{2} - e^{-1}\right) - (0-1) = \frac{5}{2} - e^{-1}$$

6.1.23.

$$\int_0^3 \int_0^{\pi/2} t^2 \sin^3\phi\, d\phi\, dt$$

$$(*) \int \sin^3\phi\, d\phi = \int \sin\phi\, (1 - \cos^2\phi)\, d\phi. \qquad u = \cos^3\phi$$

$$du = -\sin\phi\, d\phi.$$

$$= -\cos\phi + \frac{\cos^3\phi}{3} + C.$$

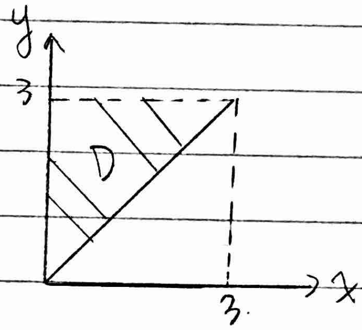
$$\int_0^{\pi/2} \sin^3\phi \, d\phi = \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/2} = (-0+0) -$$

$$(-1+1/3) = 2/3.$$

$$\int_{t=0}^{3} \frac{2}{3} t^2 \, dt = \frac{2}{9} t^3 \Big|_0^3 = 6.$$

6.2.9. $\iint_D e^{-y^2} dA$, $D = \{(x,y) \mid 0 \le y \le 3,\ 0 \le x \le y\}$.

$$= \int_0^3 \int_0^y e^{-y^2} \, dx \, dy. \quad (\text{type II})$$



$$= \int_0^3 \left[x e^{-y^2}\right]_0^y dy.$$

$$= \int_0^3 y e^{-y^2} dy. \qquad u = -y^2$$

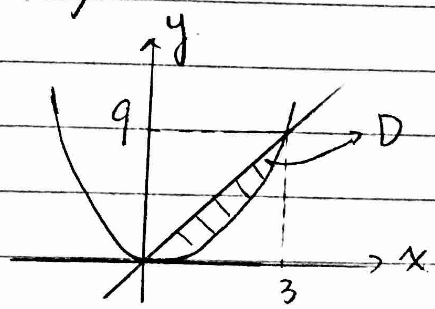
$$du = -2y \, dy.$$

$$\int y e^{-y^2} dy = -\frac{1}{2}\int e^u du = -\frac{1}{2} e^u = -\frac{1}{2} e^{-y^2}.$$

$$= \left[-\frac{1}{2} e^{-y^2}\right]_0^3 = \left(-\frac{1}{2}e^{-9}\right) - \left(-\frac{1}{2}e^0\right) = -\frac{1}{2}e^{-9} + \frac{1}{2}$$

6.2.12. $\iint_D xy \, dA$, D est borné par $y = x^2$, $y = 3x$.

$D = \{(x,y) \mid 0 \le x \le 3,\ x^2 \le y \le 3x\}$.
(type I).



$$\iint_D xy \, dA = \int_{x=0}^{3} \int_{y=x^2}^{3x} xy \, dy \, dx.$$

$$= \int_{x=0}^{3} \left[\frac{xy^2}{2}\right]_{x^2}^{3x} dx. = \int_0^3 \left(\frac{9x^2 \cdot x}{2} - \frac{x^5}{2}\right) dx =$$

$$\left[\frac{9}{2}x^4 \cdot \frac{1}{4} - \frac{x^6}{12}\right]_0^3 = \frac{9}{8}\cdot 3^4 - \frac{1}{12}\cdot 3^6 = \frac{243}{8}.$$

6.2.17. $\iint_D (2x-y)\,dA.$ $\quad D=\{(x,y) \mid x^2+y^2 \le 4\}.$

Coordonnées polaires:

$x = r\cos\theta$, $y = r\sin\theta$. , $0 \le \theta \le 2\pi$, $0 \le r \le 2$.

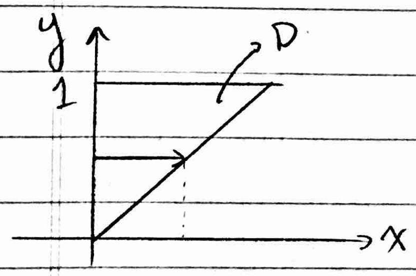
$dA = r\,dr\,d\theta$.

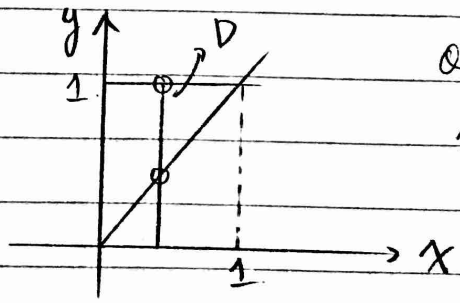
$$\int_0^{2\pi}\int_0^2 (2r\cos\theta - r\sin\theta)\, r\,dr\,d\theta.$$

$$= \int_0^{2\pi} (2\cos\theta - \sin\theta)\left(\frac{r^3}{3}\right)\Big|_{r=0}^{2} d\theta.$$

$$= \frac{8}{3}\left[2\sin\theta + \cos\theta\right]_0^{2\pi} = \frac{8}{3}\left[0+1-(0+1)\right] = 0.$$

6.2.41. $\int_0^1\int_0^y f(x,y)\,dx\,dy.$ $\quad D=\{(x,y) \mid 0 \le x \le y,\ 0 \le y \le 1\}.$





Quand $0 \le x \le 1$

$x \le y \le 1.$

$$\int_0^1\int_x^1 f(x,y)\,dy\,dx. \quad D=\{(x,y) \mid 0 \le x \le 1,\ x \le y \le 1\}.$$

6.3.17. $r = 5\cos\theta$. Son équation cartésienne ?

$r = 5\cos\theta \Rightarrow r^2 = 5r\cos\theta$.

$\begin{cases} r^2 = x^2+y^2 \\ x = r\cos\theta \end{cases} \Rightarrow x^2+y^2-5x=0. \Rightarrow \left(x-\frac{5}{2}\right)^2+y^2 = \frac{25}{4} = \left(\frac{5}{2}\right)^2.$

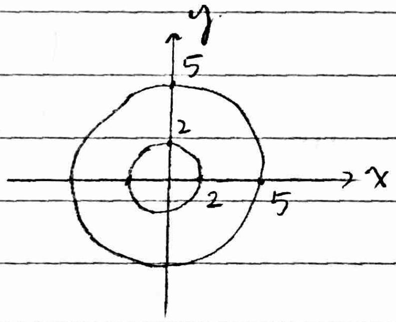
Cercle de rayon $5/2$ centré en $(5/2, 0)$.

6.3.25. $x^2+y^2=2cx$. Son équation polaire ?

$r^2 = 2c \cdot r\cos\theta$.

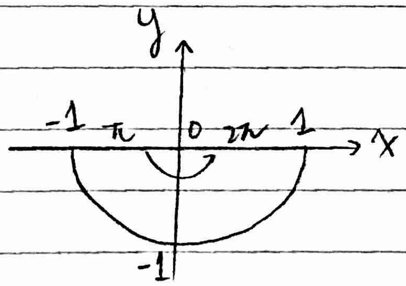
$\Rightarrow r = 2c\cos\theta$.

6.4.1.



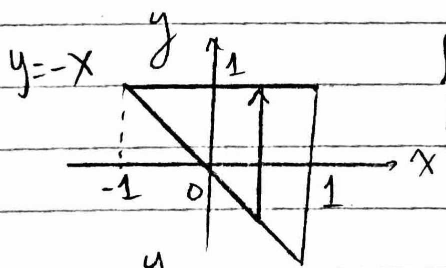
$$\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

6.4.2.



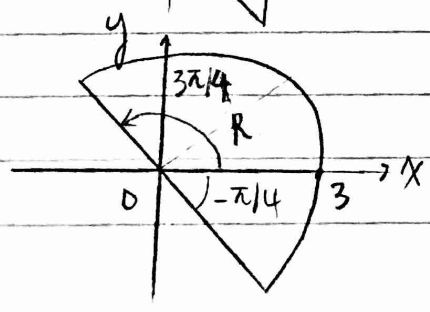
$$\iint_R f(x,y)\,dA = \int_\pi^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

6.4.3.



$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{-x}^{1} f(x,y)\,dy\,dx.$$

6.4.4



$$\iint_R f(x,y)\,dA = \int_{-\pi/4}^{3\pi/4}\int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

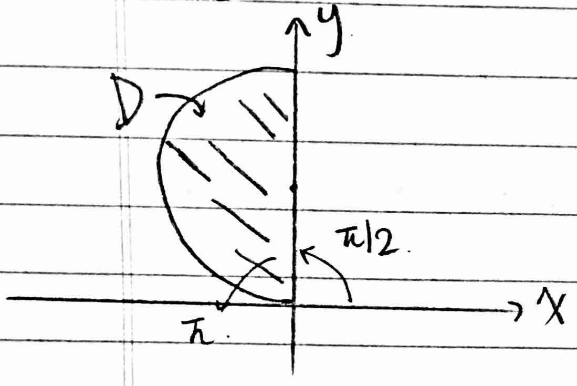
6.4.6. Esquissez la région dont l'aire est donnée par l'intégrale et calculer celle-ci.

$\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$. $\quad 0 \le r \le 2\sin\theta \Rightarrow$

$r^2 \le 2r\sin\theta \Rightarrow$

$x^2+y^2 \le 2y \Rightarrow x^2+(y-1)^2 \le 1.$

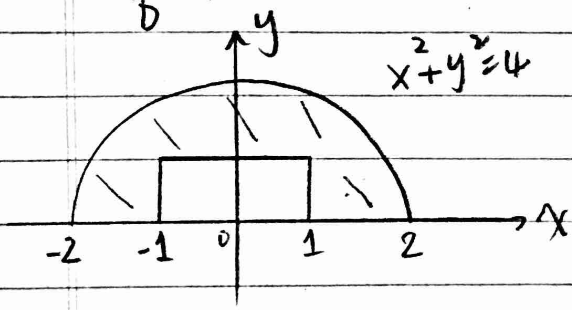
cercle rayon 1 centré en $(0,1)$



$$\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi} \frac{r^2}{2}\Big|_0^{2\sin\theta} d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} 4\sin^2\theta\, d\theta =$$

$$\int_{\pi/2}^{\pi} (1-\cos\theta)\, d\theta = \pi/2.$$

6.4.16. $\iint_D y\,dA$, où D est la région représentée.



$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \pi\}$
$\setminus \{(x,y) \mid 0 \le y \le 1,\ -1 \le x \le 1\}.$

$$\iint_D y\,dA = \int_0^{\pi}\int_0^2 r\sin\theta\, r\,dr\,d\theta - \int_{-1}^{1}\int_0^1 y\,dy\,dx.$$

$$= \int_0^{\pi}\sin\theta\,d\theta \int_0^2 r^2\,dr - \int_{-1}^1 dx \int_0^1 y\,dy$$

$$= -\cos\theta\Big|_0^{\pi}\ \frac{r^3}{3}\Big|_0^2 - x\Big|_{-1}^{1}\ \frac{y^2}{2}\Big|_0^1$$

$$= 13/3$$

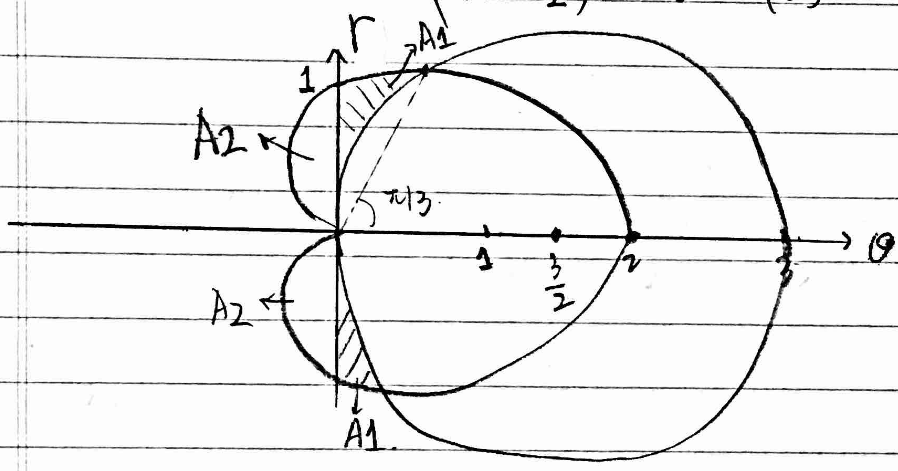
6.4.20. La région à l'intérieur de la cardioïde $r = 1 + \cos\theta$ et à l'extérieur du cercle $r = 3\cos\theta$.

$r = 1 + \cos\theta$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	2	$\frac{\sqrt{3}}{2}+1$	$\frac{\sqrt{2}}{2}+1$	$\frac{3}{2}$	1	$1/2$	$-\frac{\sqrt{2}}{2}+1$	$-\frac{\sqrt{3}}{2}+1$	0

$r = 3\cos\theta \Rightarrow r^2 = 3r\cos\theta$

$\Rightarrow x^2 + y^2 = 3x \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$



$1 + \cos\theta = 3\cos\theta$

$\to \theta = \pi/3$

$$A_1 = \int_{\pi/3}^{\pi/2} \int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta = \int_{\pi/3}^{\pi/2} \left[\frac{r^2}{2}\right]_{3\cos\theta}^{1+\cos\theta} d\theta$$

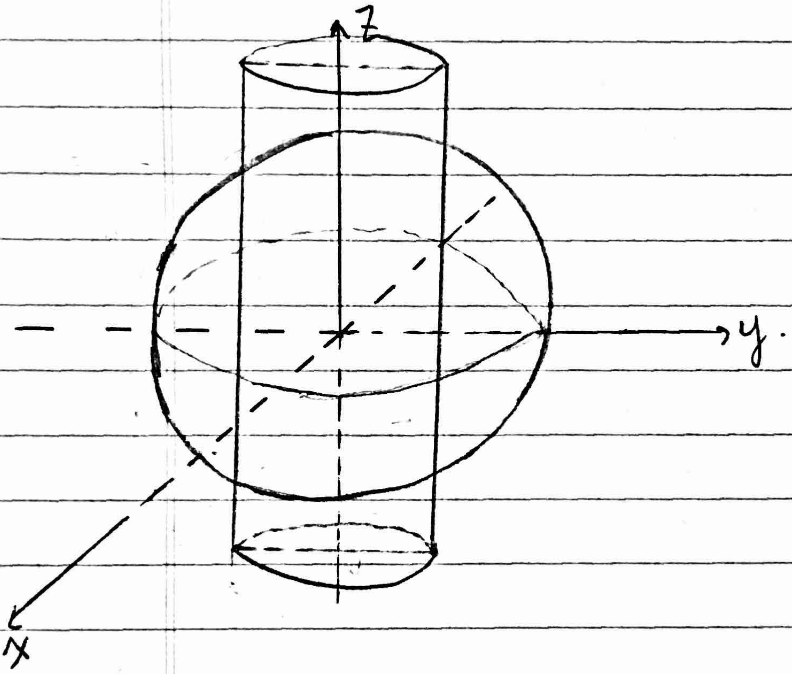
$$= \frac{1}{2} \int_{\pi/3}^{\pi/2} (1 + \cos^2\theta + 2\cos\theta - 9\cos^2\theta)\, d\theta$$

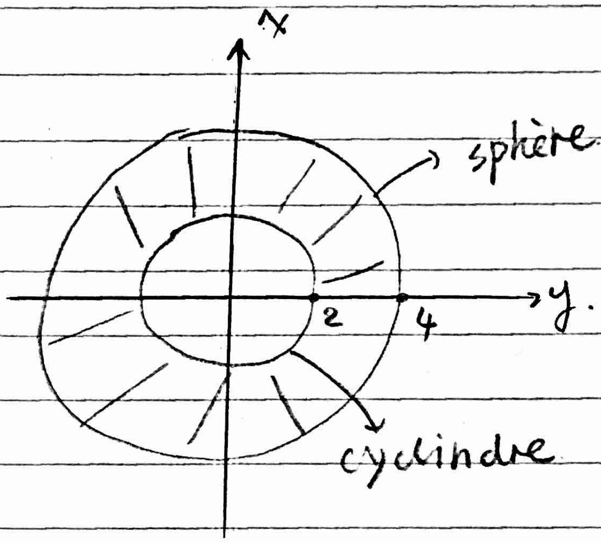
$$= 1 - \pi/4$$

$$A_2 = \int_{\pi/2}^{\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta = \frac{1}{2} \int_{\pi/2}^{\pi} (1+\cos\theta)^2\, d\theta = \frac{3\pi}{8} - 1$$

$$\text{l'aire totale} = 2(A_1 + A_2) = \frac{\pi}{4}$$

6.4.26. Calculer le volume du solide donné :
à l'intérieur de la sphère $x^2+y^2+z^2=16$ et à l'extérieur du cyclindre $x^2+y^2=4$.





$$z = \sqrt{16-x^2-y^2}$$

$$V = 2\int_{\theta=0}^{2\pi}\int_{r=2}^{4} \sqrt{16-x^2-y^2}\; r\,dr\,d\theta \qquad x^2+y^2=r^2$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=2}^{4} \sqrt{16-r^2}\; r\,dr\,d\theta$$

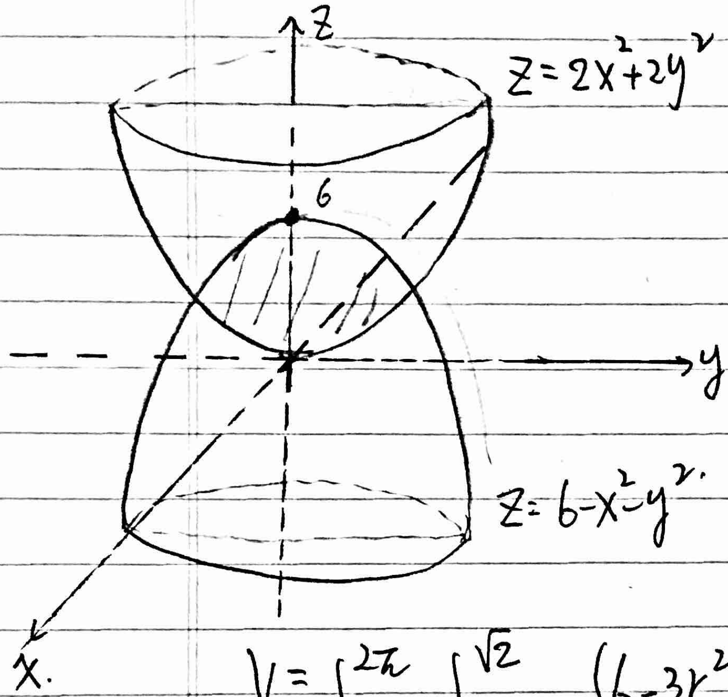
$$u = 16-r^2 \qquad du = -2r\,dr \qquad \int \sqrt{u}\; -\frac{1}{2}\,du = -\frac{1}{2}\cdot\frac{2}{3}\,u^{\frac{3}{2}}$$

$$= -\frac{1}{3}\left(16-r^2\right)^{\frac{3}{2}}$$

$$V = 2\int_0^{2\pi} d\theta \times \left[-\frac{1}{3}(16-r^2)^{\frac{3}{2}}\right]_{r=2}^{4}$$

$$= 4\pi\left(0-\left(-\frac{1}{3}\cdot 12^{\frac{3}{2}}\right)\right) = \frac{4}{3}\pi\cdot 12^{\frac{3}{2}} = 32\sqrt{3}\pi$$

6.4.30. Calculer le solide borné par $z = 6-x^2-y^2$ et $z = 2x^2+2y^2$



$$V = \iint_D (6-x^2-y^2)-(2x^2+2y^2)\,dv.$$

$x^2+y^2=r^2$.

$2x^2+2y^2 = 6-x^2-y^2$

$\Leftrightarrow\ 2r^2 = 6-r^2 \ \Leftrightarrow\ r^2=2.$

$\Leftrightarrow\ 0 \leq r \leq \sqrt{2}$

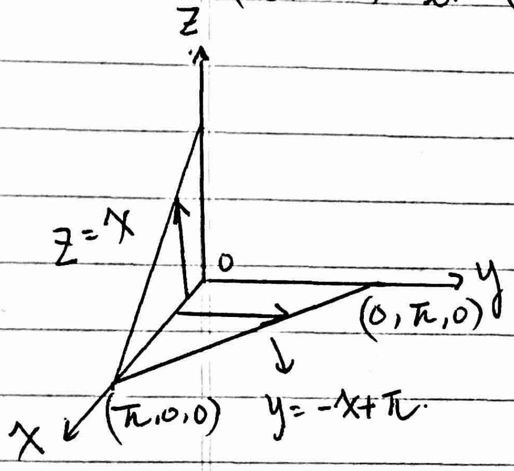
$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}} (6-3r^2)\, r\,dr\,d\theta.$$

$$= 2\pi.\left[3r^2 - \frac{3r^4}{4}\right]_0^{\sqrt{2}} = 2\pi\left(6 - \frac{3\times 2\times 2}{4}\right) = 6\pi.$$

Chapitre VII.

7.1.12. Calculez l'intégrale triple:

$\iiint_E \sin y\, dV$, où E est sous le plan $z=x$ et au-dessus de la région triangulaire dont les sommets sont $(0,0,0)$, $(\pi,0,0)$ et $(0,\pi,0)$



Quand $0\le x\le \pi$, $0\le y\le \pi-x$, $0\le z\le x$.

$$V=\int_{x=0}^{\pi}\int_{z=0}^{x}\int_{y=0}^{\pi-x}\sin y\, dy\, dz\, dx$$

$$=\int_{x=0}^{\pi}\int_{z=0}^{x}\left[-\cos y\right]_0^{\pi-x} dz\, dx$$

$$=\int_{x=0}^{\pi}\int_{z=0}^{x}\left(1-\cos(\pi-x)\right) dz\, dx$$

$$=\int_{x=0}^{\pi}\left[z-z\cos(\pi-x)\right]_{z=0}^{x} dx$$

$$=\int_{x=0}^{\pi}\left[x-x\cos(\pi-x)\right] dx$$

$$=\int_0^{\pi} x\, dx-\int_0^{\pi} x\cos(\pi-x)\, dx$$

(*). $\int x\cos(\pi-x)\, dx$ intégration par partie.

$\int uv'\, dx = uv-\int u'v\, dx$. $\quad u=x \quad v'=\cos(\pi-x)$

$\quad u'=1 \quad v=-\sin(\pi-x)$

$=-x\sin(\pi-x)-\int -\sin(\pi-x)\, dx$

$=-x\sin(\pi-x)+\cos(\pi-x) = -x\sin(\pi-x)+\cos(\pi-x)$.

$$V = \left[\frac{x^2}{2}\right]_0^{\pi} - \left[\cos(\pi - x) - x\sin(\pi - x)\right]_0^{\pi} = \frac{\pi^2}{2} - 2.$$

7.1.20. Calculez l'intégrale triple:

Le solide borné par $y = x^2 + z^2$ et $y = 8 - x^2 - z^2$.

$$V = \iiint_E \int_{x^2+z^2}^{8-x^2-z^2} dy\, dx\, dz.$$

$$8 - x^2 - z^2 = \cancel{8}\, x^2 + z^2 \Rightarrow 2(x^2+z^2) = 8.$$

$$x^2 + z^2 = r^2 \Rightarrow 2r^2 = 8 \Rightarrow r^2 = 4 \Rightarrow 0 \le r \le 2.$$

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{y=r^2}^{8-r^2} dy\, r\, dr\, d\theta.$$

$$= \int_0^{2\pi} d\theta \int_0^2 (8 - 2r^2)\, r\, dr.$$

$$= 2\pi \cdot \left[4r^2 - \frac{r^4}{2}\right]_{r=0}^{2} = 16\pi.$$

7.1.24. Si E est le prisme borné par les plans $z = c - cx$, $z = cx + c$, $z = 0$, $y = -2$ et $y = 2$, où c est une constante positive, pour quelle valeur de c le volume de E est-il égal à 8 ?

$$cx + c = -cx + c \Rightarrow x = 0.$$

$$\begin{cases} z = cx + c \\ z = 0 \end{cases} \Rightarrow c = -cx \Rightarrow x = -1.$$

$$-1 \le x \le 0, \Rightarrow z = c + cx \in [0, c + cx].$$

$$\begin{cases} z = -cx + c \\ z = 0 \end{cases} \Rightarrow c = cx \Rightarrow x = 1.$$

$$0 \le x \le 1 \Rightarrow z = c - cx \in [0, c - cx].$$

$$V=\int_{y=-2}^{2}\int_{x=-1}^{0}\int_{z=0}^{c+cx} dz\,dx\,dy+\int_{y=-2}^{2}\int_{x=0}^{1}\int_{z=0}^{c-cx} dz\,dx\,dy.$$

$$=4\cdot\int_{-1}^{0}(c+cx)\,dx+4\int_{0}^{1}(c-cx)\,dx$$

$$=4\cdot\left[cx+\frac{cx^2}{2}\right]_{-1}^{0}+4\left[cx-\frac{cx^2}{2}\right]_{0}^{1}$$

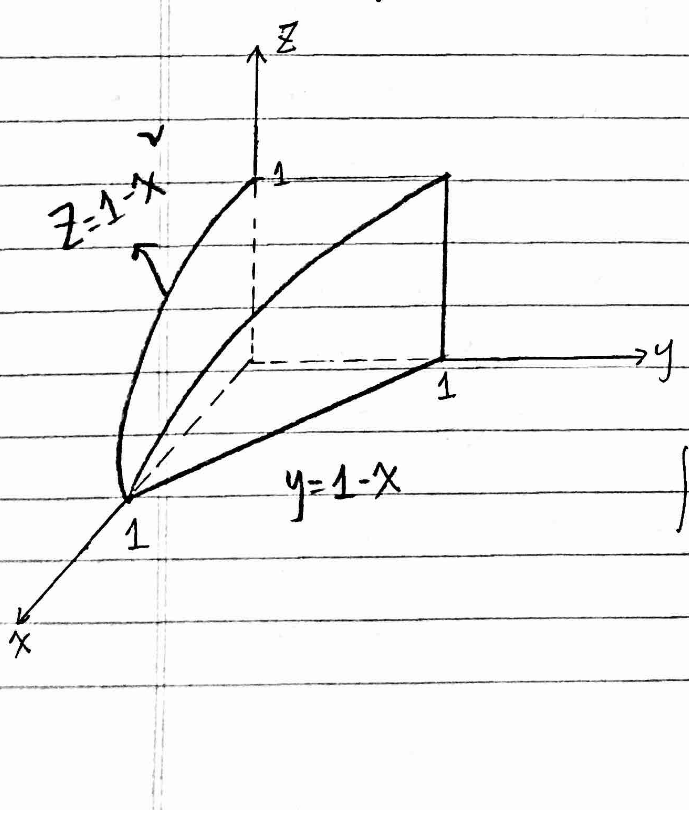
$$=4\left(-\left(-c+\frac{c}{2}\right)\right)+4\left(c-\frac{c}{2}\right)=4c.$$

posons $4c=8 \Rightarrow c=2.$

7.1.36. La figure montre le domaine d'intégration de l'intégrale

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx.$$

Réécrivez cette intégrale sous la forme d'une intégrale itérée équivalente selon les cinq autres ordres d'intégration



(1) $u_1(x,y)\le z\le u_2(x,y)$,

$D=\{(x,y)\mid 0\le x\le 1,$
$h_1(x)\le y\le h_2(x)\}.$

$\underbrace{0\le x\le 1,\quad 0\le y\le 1-x.}_{D}\quad 0\le z\le 1-x^2$

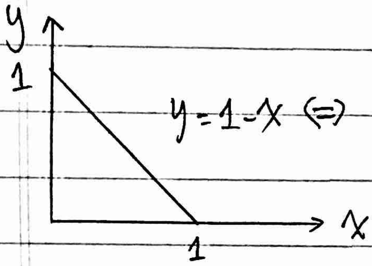
$$\int_0^1\int_0^{1-x}\int_0^{1-x^2} dz\,dy\,dx.$$

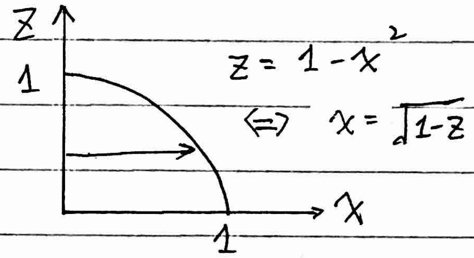
(2) $u_1(x,z) \le y \le u_2(x,z)$, $D = \{(x,z) \mid 0 \le x \le 1,\ h_1(x) \le z \le h_2(x)\}$.

$\underbrace{0 \le x \le 1, \quad 0 \le z \le 1-x^2}_{D}$ $\quad 0 \le y \le 1-x$.

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} dy\, dz\, dx.$$

(3) $u_1(x,z) \le y \le u_2(x,z)$, $D = \{(x,z) \mid 0 \le z \le 1,\ h_1(z) \le x \le h_2(z)\}$.



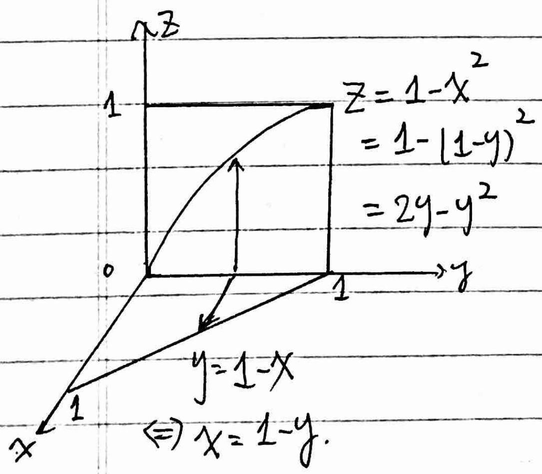


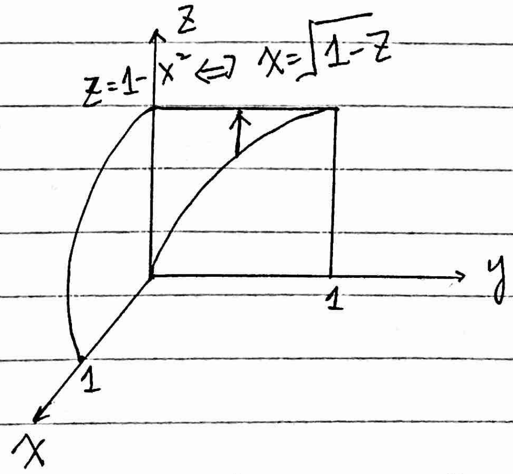
$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} dy\, dx\, dz.$$

(4) $u_1(x,y) \le z \le u_2(x,y)$, $D = \{(x,y) \mid 0 \le y \le 1,\ h_1(y) \le x \le h_2(y)\}$.

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} dz\, dx\, dy$$

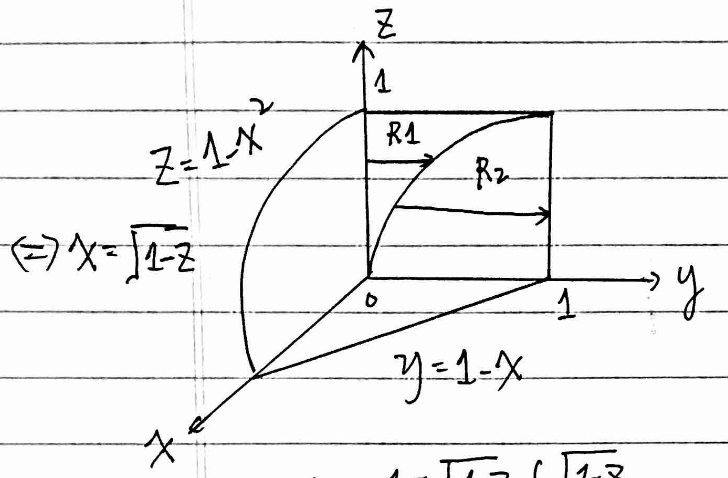
(5) $u_1(y,z) \le x \le u_2(y,z)$, $D = \{(y,z) \mid 0 \le y \le 1,\ h_2(y) \le z \le h_2(y)\}$.





$$\int_{y=0}^1 \int_{z=0}^{2y-y^2} \int_{x=0}^{1-y} dx\, dz\, dy + \int_{y=0}^1 \int_{z=2y-y^2}^{1} \int_{x=0}^{\sqrt{1-z}} dx\, dz\, dy.$$

(b) $u_1(y,z) \le x \le u_2(y,z)$, $D = \{(y,z) \mid 0 \le z \le 1,\ h_1(z) \le y \le h_2(z)\}$.



R1: $0 \le z \le 1$,
$0 \le y \le 1 - \sqrt{1-z}$.
$0 \le x \le \sqrt{1-z}$

R2: $0 \le z \le 1$
$1 - \sqrt{1-z} \le y \le 1$.
$0 \le x \le 1 - y$.

$$\int_{z=0}^{z=1} \int_{0}^{1-\sqrt{1-z}} \int_{x=0}^{\sqrt{1-z}} dx\, dy\, dz +$$

$$\int_{0}^{1} \int_{1-\sqrt{1-z}}^{1} \int_{0}^{1-y} dx\, dy\, dz.$$

7.2.8. Identifiez la surface dont l'équation donnée.

$r = 2\sin\theta$.

$r^2 = 2\sin\theta\, r. \Leftrightarrow x^2+y^2 = 2y. \Leftrightarrow x^2+(y-1)^2 = 1.$

C'est un cylindre d'axe z, de base circulaire de rayon 1.

7.2.24. Identifiez la surface dont l'équation sphérique donnée.

$\rho = \cos\phi. \Leftrightarrow \rho^2 = \cos\phi\, \rho.$

$\Leftrightarrow x^2+y^2+z^2 = z. \Leftrightarrow x^2+y^2+(z-\tfrac{1}{2})^2 = \tfrac{1}{4} = (\tfrac{1}{2})^2.$

C'est un sphère de rayon 1/2 centrée en (0, 0, 1/2)

7.3.6. Utilisez les coordonnées cyclindres.

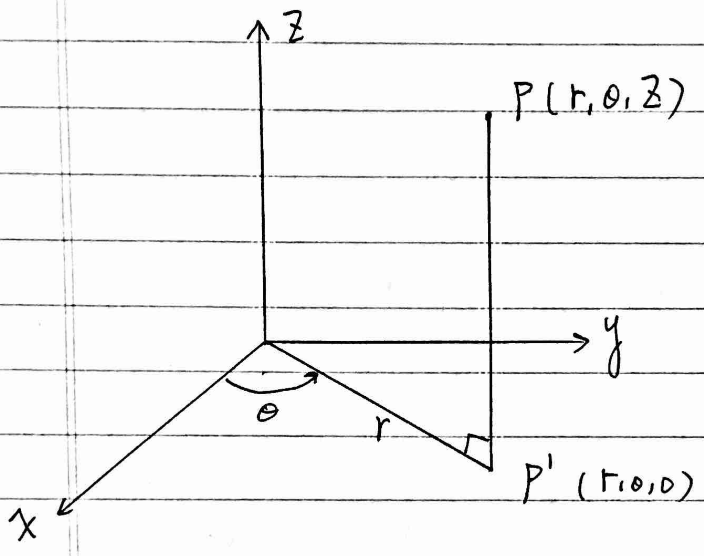
Calculez $\iiint_E (x-y)\, dV$, où E est le solide compris

entre les cyclindres $x^2+y^2=1$ et $x^2+y^2=16$,

au dessus du plan xy et sous le plan $z = y+4$.

(plan ~~x~~y : $z=0$)

Rappel: Coordonnées cylindriques:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$dv = r\,dz\,dr\,d\theta$.

$$\begin{cases} 0 \le z \le y+4 \\ 1 \le x^2+y^2 \le 4 \end{cases} \Rightarrow \begin{cases} 0 \le z \le r\sin\theta+4 \\ 1 \le r^2 \le 4 \iff 1 \le r \le 2. \end{cases}$$

$$V = \iiint_E (x-y)\, r\,dz\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{4}\int_{z=0}^{r\sin\theta+4} (r\cos\theta - r\sin\theta)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 r^2(\cos\theta-\sin\theta)(r\sin\theta+4)\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_1^4 r^2(\cos\theta\, r\sin\theta + 4\cos\theta - \sin^2\theta\, r - 4\sin\theta)\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_1^4 r^3(\cos\theta\sin\theta - \sin^2\theta)\,dr\,d\theta +$$

$$\int_0^{2\pi}\int_1^4 4r^2(\cos\theta-\sin\theta)\,dr\,d\theta.$$

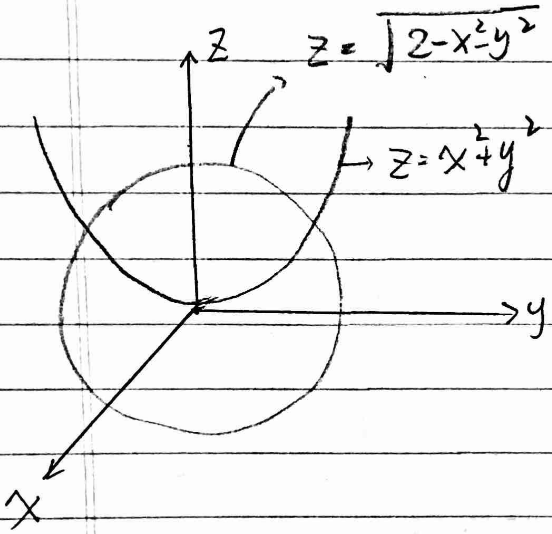
$$= \int_0^{2\pi} \frac{r^4}{4}\left(\cos\theta\sin\theta - \sin^2\theta\right)\Big|_{r=1}^{r=4} d\theta + \int_0^{2\pi} \frac{4r^3}{3}(\cos\theta-\sin\theta)\Big|_{r=1}^{4} d\theta$$

$$= (4^3 - \tfrac{1}{4}) \left(\tfrac{1}{2}\sin^2\theta - \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta \right)\Big|_{\theta=0}^{2\pi} +$$

$$\left(\frac{4^4}{3} - \frac{4}{3}\right)\left(\sin\theta + \cos\theta \right)\Big|_{\theta=0}^{2\pi}.$$

$$= -\pi (4^3 - \tfrac{1}{4})$$

7.3.10. Trouvez le volume du solide de $z = x^2 + y^2$ et à l'intérieur de la sphère $x^2+y^2+z^2 = 2$.



$$\left.\begin{aligned} z &= \sqrt{2-x^2-y^2} \\ z &= x^2+y^2 \end{aligned}\right\} \Rightarrow r^2 \leq z \leq \sqrt{2-r^2}.$$

$$x^2+y^2 = r^2. \iff z = r.$$

$$\left.\begin{aligned} z &= \sqrt{2-r^2} \\ z &= r^2. \end{aligned}\right\} \Rightarrow \begin{aligned} 2-r^2 &= r^4. \\ r^4+r^2-2 &= 0. \end{aligned}$$

$$(r^2+2)(r^2-1) = 0 \Rightarrow r^2 = 1.$$

$$0 \leq r^2 \leq 1 \Rightarrow 0 \leq r \leq 1.$$

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta.$$

$$= \int_0^{2\pi} d\theta \int_{r=0}^{1} r\left(\sqrt{2-r^2} - r^2\right) dr.$$

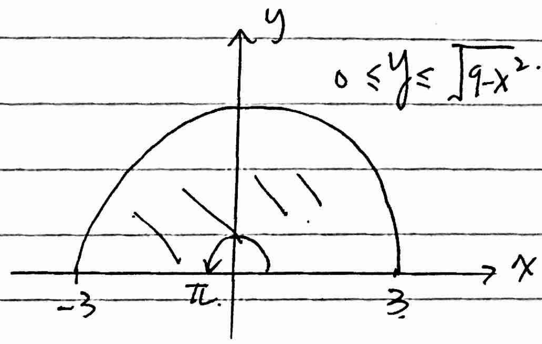
$$= 2\pi. \left[\frac{(2-r^2)^{3/2} \cdot \frac{2}{3}}{-2} - \frac{r^4}{4} \right]_{r=0}^{1}.$$

$$= 2\pi\left(-\frac{1}{3} - \frac{1}{4} + \frac{1}{3}\sqrt{8}\right).$$

7.3.18. Calculez l'intégrale en utilisant les coordonnées cylindriques.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx.$$

$0 \leq y \leq \sqrt{9-x^2}$.



$0 \leq z \leq 9-x^2-y^2 \iff 0 \leq z \leq 9-r^2.$

$0 \leq \theta \leq \pi.$

$0 \leq r \leq 3.$

$$V=\int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r^2\,dz\,dr\,d\theta.$$

$$=\int_0^{\pi}\int_0^{3} r^2(9-r^2)\,dr\,d\theta = \pi\cdot 9\cdot(9-27/5))$$

7.4.2. $$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

$$=\int_0^{2\pi}d\theta\cdot\left[\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\right].$$

$$=2\pi\cdot\left[\int_{\phi=0}^{\pi/4}\left[\frac{\rho^3\sin\phi}{3}\right]\Bigg|_{\rho}^{\sec\phi}d\phi\right].$$

$$=2\pi\left[\int_{\phi=0}^{\pi/4}\frac{\sec^3\phi\,\sin\phi}{3}\,d\phi\right]$$

$$=\frac{2\pi}{3}\cdot\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} = \pi/3.$$

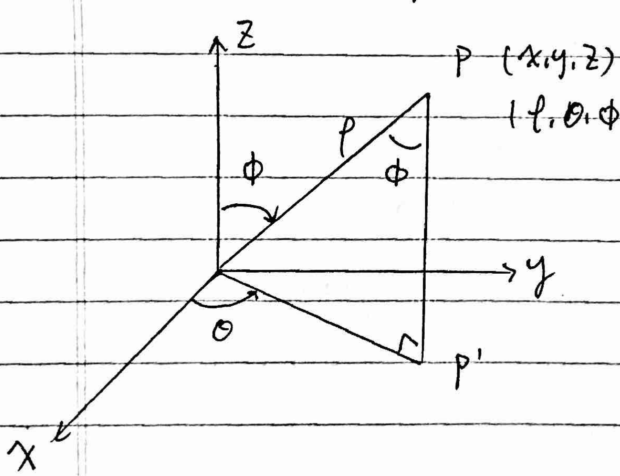
7.4.5 Calculez $\iiint_B (x^2+y^2+z^2)^2 \, dv$, où B est la boule de rayon 5 centrée à l'origine.

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{5} (\rho^2)^2 \underbrace{\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}_{dV}$$

$$= 2\pi \cdot \left(-\cos\phi\right)\Big|_0^{\pi} \; \frac{\rho^7}{4}\Big|_{\rho=0}^{5} = 4\pi \cdot \frac{5^7}{7}.$$

7.4.8. Calculez $\iiint_E y^2 \, dv$, où E est l'hémisphère solide $x^2+y^2+z^2 \leq 9$, $y \geq 0$.

Rappel: coordonnées sphériques.

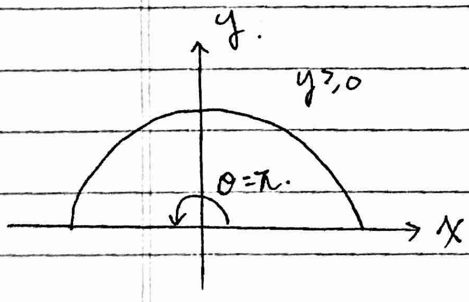


$$\rho^2 = x^2+y^2+z^2.$$

$$\begin{cases} z = \rho\cos\phi \\ r = \rho\sin\phi. \end{cases} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

$$\Rightarrow \begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi. \end{cases}$$

$0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$



$$E = \{(\rho,\theta,\phi) \mid 0 \leq \rho \leq 3, \; 0 \leq \theta \leq \pi, \; 0 \leq \phi \leq \pi\}.$$

$y = \rho\sin\phi\sin\theta.$

$$\int_{\theta=0}^{\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{3} \rho^2 \sin^2\phi \sin^2\phi \; \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$= \frac{2 \cdot 3^4}{5}\pi.$$

Calculez le volume du solide compris entre
$z = \sqrt{x^2+y^2}$ et $x^2+y^2+z^2 = 2$.

Coordonnées cylindriques:

$$\begin{cases} z = \sqrt{x^2+y^2} \\ z = \sqrt{2-x^2-y^2} \\ x^2+y^2 = r^2 \end{cases} \Rightarrow r^2 = 2 - r^2 \Rightarrow 2r^2 = 2 \Rightarrow r^2 = 1.$$

$$\Rightarrow 0 \leq r \leq 1.$$

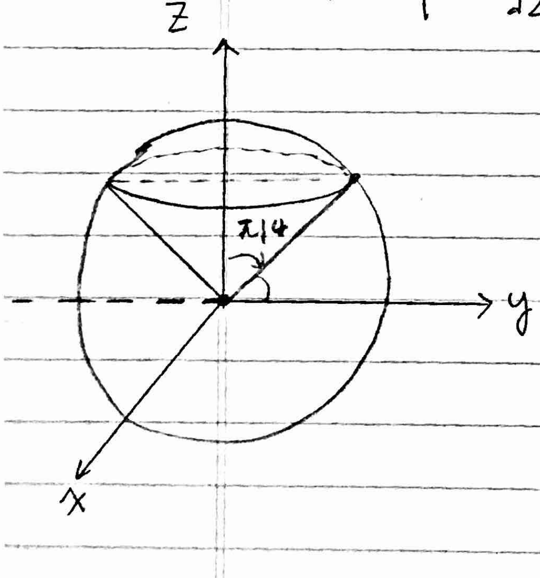
$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r}^{z=\sqrt{2-r^2}} r \, dz \, dr \, d\theta.$$

$$= 2\pi \left[\frac{2}{3}\sqrt{2} - \frac{2}{3} \right].$$

Coordonnées sphériques.

$$z = \sqrt{x^2+y^2} \Rightarrow z^2 + z^2 = 2 \Rightarrow z = 1.$$

$$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{2}} \Rightarrow \phi = \pi/4.$$



$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= 2\pi \left(\frac{2\sqrt{2}}{3} - \frac{2}{3} \right).$$